Section 3 Yuchao Yi 11/3/2026

We apply the criterion to a mixed Green kernel. Let g_1, g_2 be smooth metrics in coordinate patches centered at 0, obtained from interior patches of compact Dirichlet manifolds. Suppose they agree on an open set W containing the lower side of a smooth hypersurface through 0. After a linear coordinate change, assume

$$g_1(0) = g_2(0) = I, \qquad \{x_n < q(x')\} \subset W \text{ near } 0, \qquad q(0) = 0, \quad dq(0) = 0,$$

for a smooth function q. No regularity of the rest of ∂W is assumed. Let G_j be the Dirichlet Green kernels, normalized with metric volume, and suppose $G_1 = G_2$ on $W \times W$. For a small dilation parameter $\lambda > 0$, put

$$g_{j,\lambda}(z) = g_j(\lambda z), \qquad G_{j,\lambda}(x,y) = \lambda^{n-2} G_j(\lambda x, \lambda y), \qquad W_\lambda = \lambda^{-1} W. \tag{3.10}$$

On each fixed bounded patch the metrics tend smoothly to the Euclidean metric, and W_λ contains every fixed compact subset of $\{z_n < 0\}$ for sufficiently small λ. Initially define, off the diagonal,

$$\mathcal{G}_\lambda(x,y) = \begin{cases} G_{1,\lambda}(x,y), & y \in W_\lambda, \\ G_{2,\lambda}(x,y), & x \in W_\lambda. \end{cases} \tag{3.11}$$

The pieces fit and solve the first equation in x and the second in y.

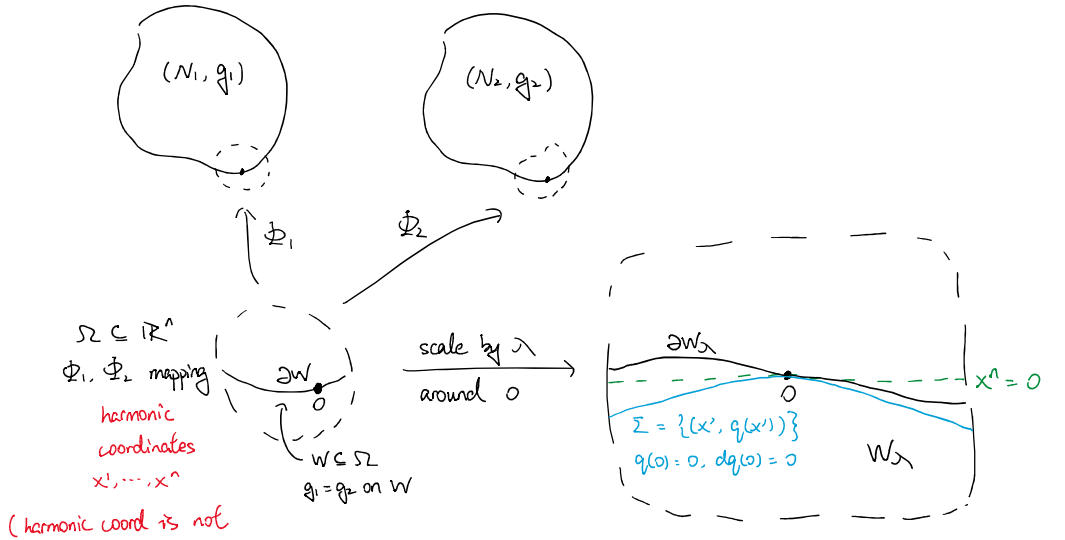
Proposition 3.5 (Fixed-separation initialization). *In this setting, let $n \geq 3$ and fix $B > 0$, $d > 0$ and $\eta > 0$. For all sufficiently small λ, the data (3.11) have a product-solution continuation to a neighborhood of*

$$\{(x,y) : |x|, |y| \leq B, \ |x - y| \geq d\}.$$

Every fixed C^m norm on a smaller neighborhood is bounded uniformly in λ. The continuation agrees with $G_{1,\lambda}$ where $y_n < -\eta$ and with $G_{2,\lambda}$ where $x_n < -\eta$, on the corresponding overlaps. The smallness threshold for λ and the constants may depend on B, d, η, m.

$\Delta_{g_1,\lambda,x} G_\lambda = 0$

$\Delta_{g_2,\lambda,y} G_\lambda = 0$

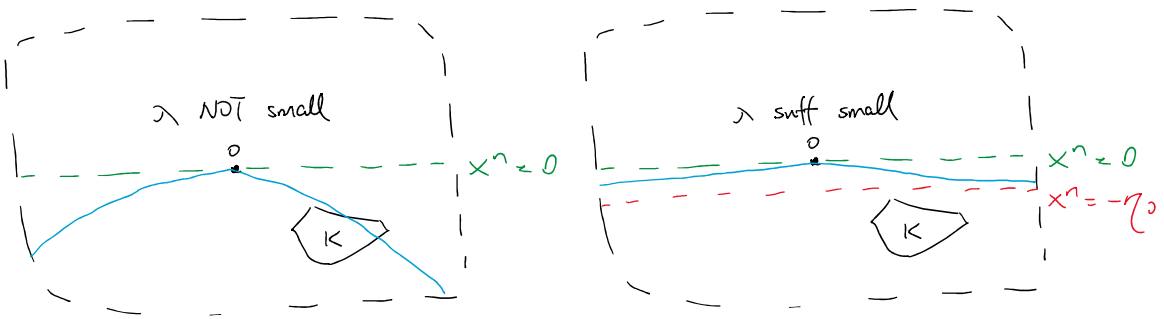


(harmonic coord is not used in Prop 3.5 but will be used later)

Σ, ∂W_λ and $\{x^n = 0\}$ are generally different things here

∂W_λ is allowed to be singular, q_λ is used as regular replacement of ∂W_λ

$\{x : x^n < q_\lambda(x')\} \subseteq W_\lambda$ implies :



$$\forall K \Subset \{x_n < 0\}, \quad \exists \lambda_0(K) > 0 : K \subset W_\lambda \quad (0 < \lambda < \lambda_0(K)).$$

In particular, for λ suff small : $\{x^n < -\eta_0, |x'| < R\} \subseteq W_\lambda$

(I) Obtain Common Seed Region $S = \{(x,y) : \min(x^n, y^n) < c_0|x-y|, \ |x|, |y| < R, \ d_0 < |x-y| < D_0\}$
for any R, d_0, D_0 prescribed and λ suff small

Lemma 3.2 (Extension from a cross). *Let D_i be bounded connected coordinate domains and $S_i \Subset D_i$ nonempty open sets. Suppose a product solution F is given consistently on*

$$(D_1 \times S_2) \cup (S_1 \times D_2),$$

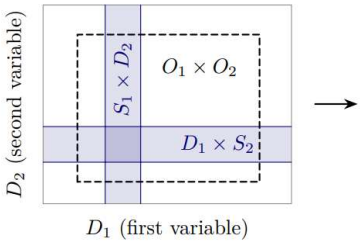
elliptic

with the sum of its two L^2 norms at most M. Let $O_i \Subset D_i$ be open sets such that every L_i-solution w on D_i satisfies (three spheres inequality)

$$\|w\|_{L^2(O_i)} \le C\|w\|^{\gamma_i}_{L^2(S_i)}\|w\|^{1-\gamma_i}_{L^2(D_i)}, \qquad 0 < \gamma_i < 1, \qquad \gamma_1 + \gamma_2 > 1. \tag{3.1}$$

Then F extends as a product solution to $O_1 \times O_2$, agreeing with both given pieces on their intersections. On every compact subset of this product its C^m norm is at most $C_m M$. The constants are uniform when the geometric margins, ellipticity, coefficient bounds, interpolation constants and positive exponent margin are uniform. Each specified output order uses only finitely many coefficient bounds.

Figure 2 summarizes the sets and the two estimates in Lemma 3.2.



Compatible solutions on the cross

$\|w\|_{O_i} \le C\|w\|_{S_i}^{\gamma_i}\|w\|_{D_i}^{1-\gamma_i}$

$0 < \gamma_i < 1, \qquad \gamma_1 + \gamma_2 > 1$

The cross series constructs a solution on the smaller product $O_1 \times O_2$.

① Use ball + annulus to show extension is possible

by Lemma 3.2, thus interpolation inequality is required

One variable is extended by annulus which may have small γ,

the other one is then extended by ball which has γ close to 1

② Show extension is unique

Since one extension is by ball, another by annulus, we may have:

a) same orientation: ball ∩ ball, annulus ∩ annulus

b) opposite orientation: ball ∩ annulus, annulus ∩ ball

Need to show overlapping region is connected to known region, then use UCP

③ Show the extension covers desired region $\{\min(x^n, y^n) < c_0|x-y|\}$

Can always assume $y^n \ge x^n$, then put y in annulus and show x is in the ball

Important: $x^n < 0$ doesn't mean $x \in W_\lambda$! Only when $x^n < -\eta_0 < 0$,

we can say $x \in W_\lambda$ for λ suff small!

B A
(x, y)

$\tilde{B}$ $\tilde{A}$
$(\tilde{x}, \tilde{y})$

$\tilde{A}$ $\tilde{B}$
$(\tilde{x}, \tilde{y})$

① $p = (p', 0)$ (p, p) $G_\lambda(x, y)$

(1)

Choose a small $\tau_0 > 0$, say $\tau_0 = 1/10$, and put

$$a = p - \tau_0 s e_n.$$

First-variable domains:

$$\boxed{\begin{aligned} D_1 &= B(a, 3\tau_0 s), \\ S_1 &= B(a, (1-\alpha)\tau_0 s), \\ O_1 &= B(a, (\tau_0 + c)s). \end{aligned}} \quad \begin{aligned} &= B(a, \tau_0 S - \alpha\tau_0 S) \\ &= B(a, \tau_0 S + cS) \end{aligned} \qquad (2)$$

(3)

based on γ_2, make α, c small then $S_1 \approx O_1$ so γ_1 almost 1

Here $\alpha, c > 0$ are small constants to be chosen later.

$\Rightarrow \gamma_1 + \gamma_2 > 1$

Second-variable domains:

Choose this first! (2) →

$$\boxed{\begin{aligned} D_2 &= \left\{ y \in \mathbb{R}^n : \frac{s}{2} < |y - p| < 2s \right\}, \\ S_2 &= B(p - se_n, s/10), \\ O_2 &= \left\{ y \in \mathbb{R}^n : \frac{3s}{4} < |y - p| < \frac{5s}{4} \right\}. \end{aligned}} \quad \text{gives } \gamma_2 \qquad (3)$$

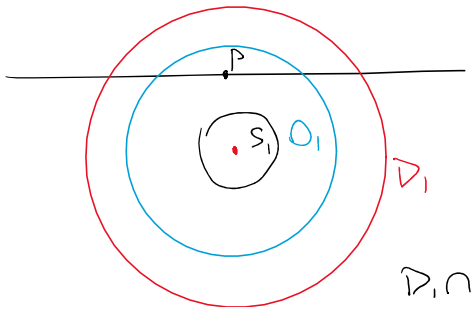
↓

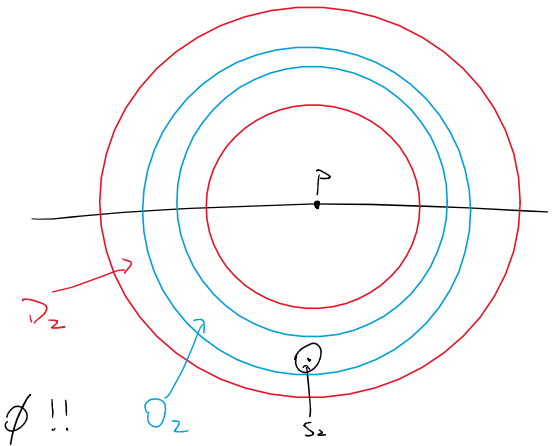
Here D_2 and O_2 are concentric annuli centered at p, while S_2 is a ball near the bottom of these annuli.

$S_1 \approx O_1$

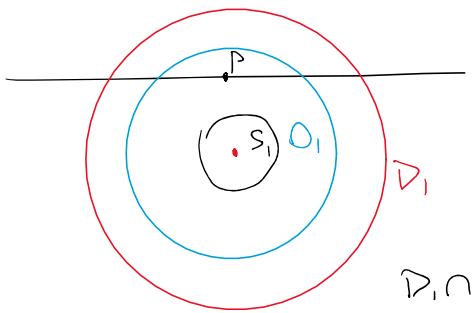
$S_1 \subseteq \{x^n < -\eta_0\}$

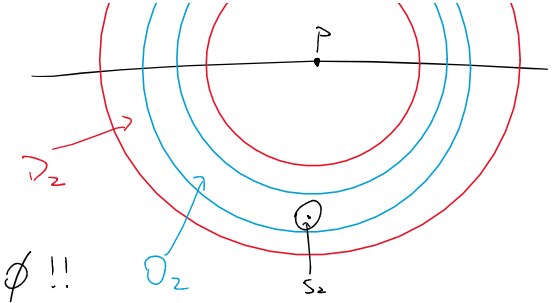
$O_1 \cap \{x^n > 0\} \neq \emptyset$





$D_1 \cap D_2 = \emptyset$!!





$D_1 \cap D_2 = \emptyset$!!

$x \in O_1,\ y \in O_2 \Rightarrow |x-y| > \frac{3s}{4} - (\tau_0 + c)s$

Need $\frac{3s}{4} - (\tau_0 + c)s > d_0 \Rightarrow$ Let c small, then $s > s_{min} = 2d_0$ enough

② Need to show extensions agree on overlapping

same orientation:

$O_1, \widetilde{O}_1$ ball ; $O_2, \widetilde{O}_2$ annulus

opposite orientation:

$O_1, \widetilde{O}_2$ ball ; $O_2, \widetilde{O}_1$ annulus

Show overlapping region is connected

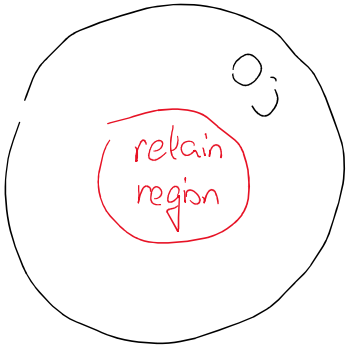
to $W_\lambda \times D_2 \cup D_1 \times W_\lambda$

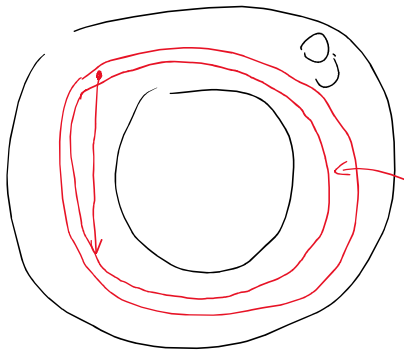
Then use UCP of $\Delta_{g_1,\lambda} x + \Delta_{g_2,\lambda} y$

(Recall $W_\lambda \supseteq \{x^n < -\eta_0\}$ so we need to move to sufficiently negative region)

K related to size of retain region

Choose it suff small here





p $Q_p = O_1 \times O_2$ (ball × annulus)

q $Q_q = \widetilde{O}_1 \times \widetilde{O}_2$

$(x,y) \in Q_p \cap Q_q$

$(x,y) \xrightarrow{\text{path}} \text{known region } Q_p \cap Q_q$

$L_{1,x} G_\lambda = 0,\ L_{2,y} G_\lambda = 0$

$(L_{1,x} + L_{2,y}) G_\lambda = 0$

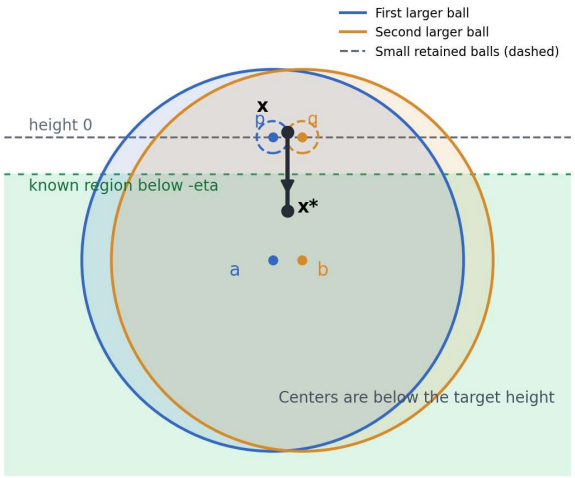
K related to size of retain region
Choose it suff small here

$L_{1,x} G_x = 0, \quad L_{2,y} G_x = 0$

$(L_{1,x} + L_{2,y}) G_x = 0$

elliptic

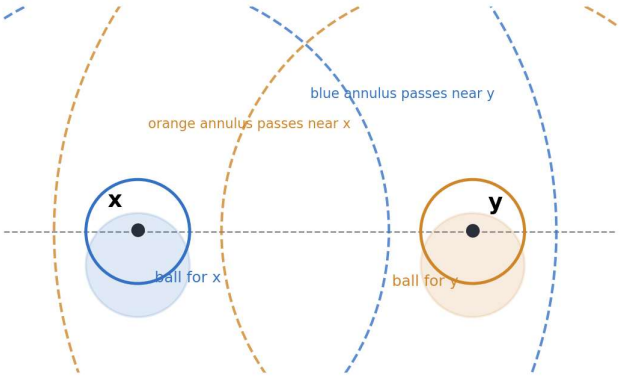
Same orientation: moving x downward inside two overlapping output balls



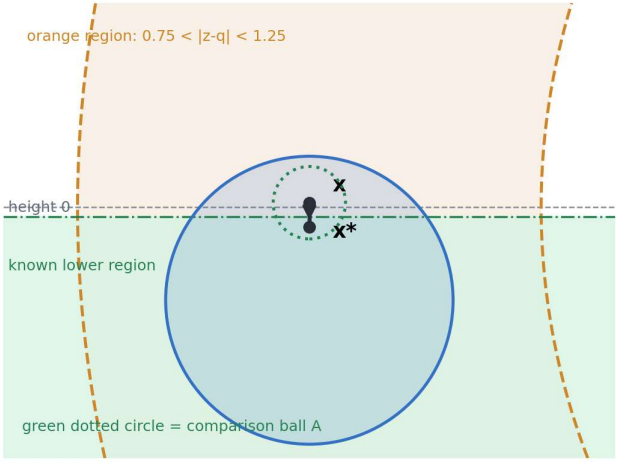
$x \in \text{ball}_p \cap \text{ball}_q$

$y \in \text{Ann}_p \cap \text{Ann}_q$

Opposite orientations: two product domains

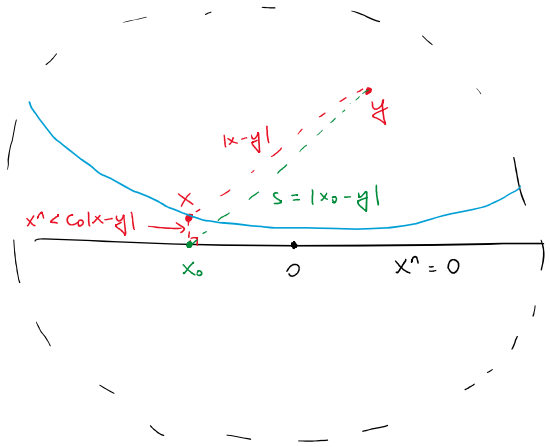


Fix y: compare by moving x inside both outputs



(3) Show retain region covers common seed $\{(x,y): \min(x^\wedge, y^\wedge) \leq C_0|x-y|\}$

Assume $y_n > x_n$



For fixed y
x below blue line
satisfies $x^n < C_0|x-y|$

$s \geq |x-y| \geq d$

$s \leq |x-y| + x_n < (1+C_0)|x-y| \leq 2B(1+C_0)$ $\Rightarrow d \leq s < 2B(1+C_0)$

Choose $C_0 < K$, then $x \in B(x_0, Ks)$ b/c $x^n < C_0|x-y| \leq C_0 s < Ks$

$\widetilde{O_1} = B(x_0, Ks) \subseteq B(x_0 - \tau_0 s e_n, (\tau_0 + C)s) = O_1$

Ks above ground $<$ Cs above ground

Since $|x_0 - y| = s$, $y \in \sim s$ annulus at x_0 $= \widetilde{O_2} \subseteq O_2$

(Similar argument for $-\eta_0 < x^n < 0$)

Choose η_0 :

Reason	Required inequality
S_1 lies below $-\eta_0$	$\eta_0 < \alpha\tau_0 d_0/2$
Gluing paths reach the known region	$\eta_0 < \kappa d_0/2$
Coverage of the common seed	$\eta_0 < c_0 d_0$
Agreement with the original kernel at depth $-\eta$	Convenient to have $\eta_0 < \eta$

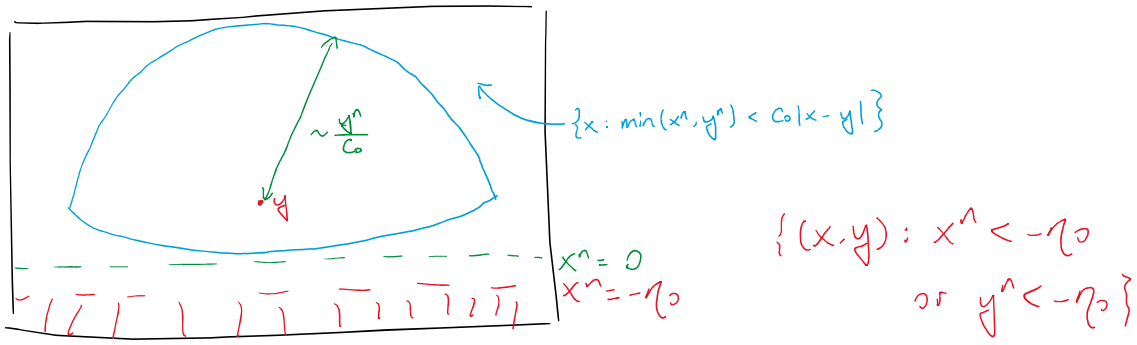
We can satisfy everything by choosing

$$\boxed{\eta_0 = \frac{1}{4}\min\{\eta,\ \alpha\tau_0 d_0,\ \kappa d_0,\ c_0 d_0\}.}$$

Since $\alpha, \tau_0, \kappa, c_0$ are universal fixed constants, the essential dependence is

$$\boxed{\eta_0 = \eta_0(d_0, \eta).}$$

Again, η_0 is fixed before λ.



$\{(x,y) : x^n < -\eta_0$ or $y^n < -\eta_0\}$

Hence for sphere of sufficiently large radius centered at y,
it is in $\{x : \min(x^n, y^n) < c_0|x-y|\}$

Starting with $B, d, \eta > 0$ from Proposition 3.5, the order for the common-seed construction is

$$\boxed{\tau_0 \longrightarrow \gamma_2 \longrightarrow (\alpha, c) \longrightarrow \kappa \longrightarrow c_0 \longrightarrow (R, d_0, D_0) \longrightarrow \eta_0 \longrightarrow \lambda.}$$

Here R, d_0, D_0 specify the bounded region and positive separation range on which we construct the common seed.

τ_0 : chosen for D_1, D_2 separate

γ_2 : interpolation inequality for annulus part

(α, c) : make O_1, S_1 suff close so $\gamma_1 + \gamma_2 > 1$

κ : size of retain region, make gluing argument easier by forcing large overlapping

c_0 : $\min(x^n, y^n) \le c_0 |x - y|$

η_0 : how deep W_λ is

λ : scaling

(II) Extend from common seed region using sweeping tubes

Proposition 3.3 (Product extension criterion). *Let ρ be smooth near $z_0 = (x_0, y_0)$, with $\rho(z_0) = 0$ and $\mathrm{d}_x\rho, \mathrm{d}_y\rho \neq 0$. Put*

$$a_i = \frac{\nabla_i \rho}{|\nabla_i \rho|^2}, \qquad H = \operatorname{Hess} \rho.$$

Suppose that at z_0

$$H((a_1, -a_2), (a_1, -a_2)) + H((v_1, v_2), (v_1, v_2)) > 0 \quad (v_i \perp a_i,\ |v_i| = |a_i|). \tag{3.7}$$

Every product solution given on $\{\rho > 0\}$ near z_0 extends to a neighborhood of z_0, agreeing on a smaller part of that side. In fact, the construction uses only two compact product sets in $\{\rho > \epsilon\}$ for some $\epsilon > 0$. The neighborhoods and estimates on smaller neighborhoods are uniform under small smooth perturbations preserving the strict hypotheses, provided the data on those compact sets are bounded.

- $\operatorname{Hess}\theta\left(\frac{\nabla\theta}{|\nabla\theta|}, \frac{\nabla\theta}{|\nabla\theta|}\right) + \operatorname{Hess}\theta(v, v) > 0$ is the psendoconvexity for laplacian
- $|a_1 + iv_1|_{g_1} = |-a_2 + iv_2|_{g_2} = 0$ in complex char set

- $\mathrm{Hess}\,\theta(\overline{|\nabla\theta|}, |\nabla\theta|) + \mathrm{Hess}\,\theta(v,v) > 0$ is the pseudoconvexity for laplacian
- $|a_1 + iv_1|_{g_1} = |-a_2 + iv_2|_{g_2} = 0$ in complex char set
- $(a_1 + iv_1, -a_2 + iv_2)$ tangent to $\{\rho = 0\}$

Let $Z = (a_1 + iv_1, -a_2 + iv_2)$

$$H(Z, \overline{Z}) \underset{\uparrow}{=} H((a_1, -a_2), (a_1, -a_2)) + H((v_1, v_2), (v_1, v_2)) > 0$$

b/c H is real symmetric

This is NOT the standard elliptic pseudoconvexity for $L_{1,x} + L_{2,y}$!!

(which is $\mathrm{Hess}_{g_1 \oplus g_2}\rho(E, E) + \mathrm{Hess}_{g_1 \oplus g_2}\rho(V, V) > 0$

where E normal to $\{\rho = 0\}$ and V tangent to $\{\rho = 0\}$,

here both $(a_1, -a_2)$ and (v_1, v_2) tangent to $\{\rho = 0\}$)

<u>Ex</u>: $g_1 = g_2 = g_{Euc}$

$$\rho = x^n + y^n + \frac{\varepsilon}{2}(|x'|^2 + |y'|^2) - \frac{M}{2}(x^n + y^n)^2, \quad M \gg \varepsilon$$

Then $(a_1, -a_2) = (e_n, -e_n)$

Since $x^n + y^n$ stays constant along $(e_n, -e_n)$

$$H((e_n, -e_n), (e_n, -e_n)) = 0$$

For any $v_i \perp e_n$, $|v_i| = 1$,

$$H((v_1, v_2), (v_1, v_2)) = 2\varepsilon > 0$$

So proposition 3.3 works here

$$H(\tfrac{1}{2}(e_n, e_n), \tfrac{1}{2}(e_n, e_n)) + H((e_n, 0), (e_n, 0)) = \varepsilon - 2M < 0$$

So proposition 3.3 works here

However : $H\left(\left(\frac{1}{\sqrt{2}}(e_n, e_n), \frac{1}{\sqrt{2}}(e_n, e_n)\right) + H\left((e_1, 0), (e_1, 0)\right)\right) = \varepsilon - 2M < 0$

so standard pseudoconvexity for Δ_{2n} doesn't hold

Standard convexification for any hypersurface :

Given ρ, consider $\rho_K = \rho + \frac{K}{2}\rho^2$

$H_{\rho_K} = H_\rho + K\, d\rho \otimes d\rho$ at $\{\rho = 0\}$

Then $H_{\rho_K}(E, E) + H_{\rho_K}(V, V)$

$= H_\rho(E, E) + H_\rho(V, V) + K|d\rho|^2$

Can always make K large s.t. standard elliptic pseudoconvexity holds

But $H_{\rho_K}\left((e_n, -e_n), (e_n, -e_n)\right) + H_{\rho_K}\left((v_1, v_2), (v_1, v_2)\right)$

$= H_\rho\left((e_n, -e_n), (e_n, -e_n)\right) + H_\rho\left((v_1, v_2), (v_1, v_2)\right)$

b/c both tangential vectors !!

One can not convexify it to satisfy product pseudoconvexity

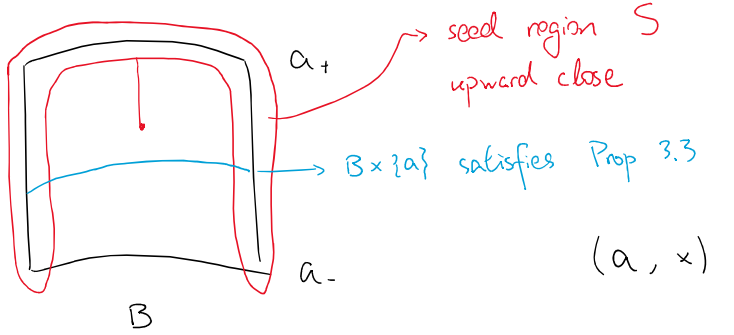
Standard elliptic pseudoconvexity for $L_{1,x} + L_{2,y}$ is not enough

b/c we need extension to each satisfy $L_{1,x} G = 0$, $L_{2,y} G = 0$

Lemma 3.4 (Propagation through a compact cylinder). *Suppose a region in a product patch has smooth coordinates (z, a) near $B \times [a_-, a_+]$, where B is a compact base, possibly with boundary or corners. A product solution is given on a neighborhood of the top and lateral boundary and on an open set $\mathcal{S}$. Assume that $\mathcal{S}$ is upward closed in a at fixed z and contains these boundary neighborhoods. If every level $a =$ constant satisfies Proposition 3.3 outside $\mathcal{S}$, with the positive side pointing toward increasing a, the solution extends to a neighborhood of the cylinder, agreeing on $\mathcal{S}$. For fixed coordinates, compact sets and strict reference inequalities, the extension has uniform compact-subset bounds under small smooth perturbations, in terms of bounds on finitely many compact subsets of the initially given region.*

B $2n-1$

$B \times [a_-, a_+] \subseteq N^2$



top: $\{a_+\} \times B$

lateral: $[a_-, a_+] \times \partial B$

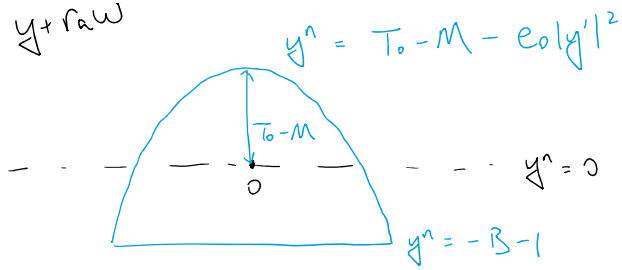
(a, x) $(\tilde{a}, x)$ $\tilde{a} > a$

Back to Prop 3.5, we need to understand what is the base and seed

Define $r_a(y) = a\, t(y)^{1+\sigma}$, where $t(y) = M + y^n + e_0 |y'|^2$, σ, e_0 small chosen later, $M > B+3$

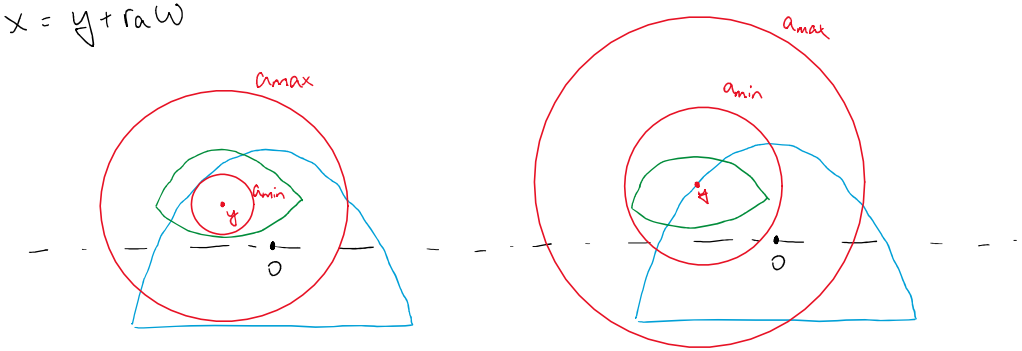
$B = \{(y, \omega) : \omega \in S^{n-1},\ y^n \geq -B-1,\ t(y) \leq T_0\}$, T_0 chosen later

$x = y + r_a \omega$



compact b/c of $e_0|y'|^2$

$x = y + r_a(y)\,\omega$



Three boundaries:

Top: $a = a_{max}$, it is in common seed region S when a_{max} large

Lateral: $y^n = -B-1$: it is in S regardless of a b/c $y^n < -\eta_0$

$y^n = T_0 - M - e_0|y'|^2$: Since $x = y + r_a(y)\,\omega = y + a t^{1+\sigma}\omega$

$\Rightarrow x^n = y^n + a t^{1+\sigma}\omega_n$

Then $\min(x^n, y^n) = y^n + \min(a t^{1+\sigma}\omega_n, 0)$

$= T_0 - M - e_0|y'|^2 + a T_0^{1+\sigma}\min(\omega_n, 0)$

$< T_0$

Suffice to require $T_0 \le c_0|x-y| = c_0 a T_0^{1+\sigma}$

Note if $\sigma = 0$, then we need $1 \le c_0 a$ which requires a large

Then for any fixed a_{min}, can take T_0 suff large for this to hold

Also, if $(x = y + r_a(y)w,\ y) \in S$, then $\forall\ \tilde{a} \in (a, a_{max}]$, $(\tilde{x} = y + r_{\tilde{a}}(y)w,\ y) \in S$

Next, need to show all level set $\{\rho_a(x,y) := \frac{1}{2}(|x-y|^2 - r_a(y)^2) = 0\}$ satisfies product pseudoconvexity

Recall product pseudoconvexity is $D^2\rho[(a_1,-a_2),(a_1,-a_2)] + D^2\rho[(v_1,v_2),(v_1,v_2)] > 0$,

where $a_i = \frac{\nabla_i \rho}{|\nabla \rho|^2}$, $|v_i| = |a_i|$, $v_i \perp a_i$

$D^2\rho[(X,Y),(X,Y)] = |X-Y|^2 - (\nabla_y r \cdot Y)^2 - r D_y^2 r[Y,Y]$ (a is omitted)

$a_1 = \frac{\nabla_x \rho}{|\nabla_x \rho|^2} = \frac{\omega}{r}$, $a_2 = \frac{\nabla_y \rho}{|\nabla_y \rho|^2} = -\frac{\omega + \nabla_y r}{r|\omega + \nabla_y r|^2}$, $\omega + \nabla_y r \neq 0$ b/c →

Consider $U_1 = r\cdot(a_1 + iv_1) = \omega + irv_1$, $U_2 = r\cdot(-a_2 + iv_2) = \frac{\omega + \nabla_y r}{|\omega + \nabla_y r|^2} + irv_2$

note $|v_1| = \frac{1}{r}$, $|v_2| = \frac{1}{r|\omega + \nabla_y r|}$, $v_1 \cdot \omega = 0$, $v_2 \cdot (\omega + \nabla_y r) = 0$

denote $V_i = rv_i$, $|V_1| = 1$, $|V_2| = \frac{1}{|\omega + \nabla_y r|}$

Plug in: we need $|U_1 - U_2|^2 - |\nabla_y r \cdot U_2|^2 - r D_y^2 r[U_2, U_2] > 0$

Note that $\omega \cdot (U_1 - U_2) = 1 - \omega \cdot U_2 = (\omega + \nabla_y r)\cdot U_2 - \omega \cdot U_2 = \nabla_y r \cdot U_2$

b/c $\omega \cdot U_1 = 1$, $(\omega + \nabla_y r)\cdot U_2 = 1$

Thus inequality reduces to

$$|\Pi_{\omega^\perp}(U_1 - U_2)|^2 - r D_y^2 r[U_2, U_2] > 0$$

(Note if r independent of y, then it reduces to $|\Pi_{\omega^\perp}(U_1 - U_2)|^2 > 0$

however $a_1 = -a_2 = \frac{\omega}{r}$, meaning we may pick $v_1 = v_2$

$\Rightarrow U_1 = U_2$ and inequality doesn't hold !!)

When is $|\Pi_{\omega^\perp}(U_1 - U_2)| = 0$:

if $\omega + \nabla_y r = 0$, recall $r_a(y) = a\,t(y)^{1+\sigma}$

then $1 = |\omega| = |\nabla_y r| = (1+\sigma)\frac{r}{t}|\nabla_y t|$

$\Rightarrow (1+\sigma)\frac{r}{t} = \frac{1}{|\nabla_y t|}$

since $\nabla_y t = (2e_0 y', 1)$

$\omega_n = -(\nabla_y r)_n = -(1+\sigma)\frac{r}{t} = -\frac{1}{|\nabla_y t|}$

$\Rightarrow \frac{x^n}{r} = \frac{y^n}{r} + \omega_n$

$< \frac{t}{r} - \frac{1}{|\nabla_y t|}$ b/c $t = M + y^n + e_0|y'|^2 > y^n$

$= (1+\sigma)|\nabla_y t| - \frac{1}{|\nabla_y t|}$

For suff small σ, e_0, we have $|\nabla_y t| \approx 1$

so $\frac{x^n}{r} < (1+\sigma)|\nabla_y t| - \frac{1}{|\nabla_y t|} \approx \sigma < c_0$

$\Rightarrow \min(x^n, y^n) \leq x^n < c_0 r = c_0|x-y|$

Hence outside S, $\omega + \nabla_y r \neq 0$

$$U_1 - U_2 = \omega - \frac{\omega + \nabla_y r}{|\omega + \nabla_y r|^2} + i(V_1 - V_2)$$

$$\overline{\Pi}_{\omega^\perp}(U_1 - U_2) = -\frac{\overline{\Pi}_{\omega^\perp}\nabla_y r}{|\omega + \nabla_y r|^2} + i\,\overline{\Pi}_{\omega^\perp}(V_1 - V_2)$$

Set to 0 obtains : ① $\nabla_y r = k\omega$

② $\overline{\Pi}_{\omega^\perp}(V_1 - V_2) = 0$

Since $V_1 \cdot \omega = 0$, $V_2 \cdot (\omega + \nabla_y r) = 0$

① $\Rightarrow \omega + \nabla_y r = (1+k)\omega \Rightarrow V_2 \cdot \omega = 0$

② $\Rightarrow V_1 = V_2$

$\Rightarrow 1 = |V_1| = |V_2| = \frac{1}{|\omega + \nabla_y r|} = \frac{1}{|1+k|}$

$\Rightarrow \nabla_y r = 0$ or $\nabla_y r = -2\omega$

Near $\nabla_y r = -2\omega$: (argument similar to $\omega + \nabla_y r \neq 0$)

$$2 = |-2\omega| = |\nabla_y r| = (1+\sigma)\frac{r}{t}|\nabla_y t|$$

$$\Rightarrow (1+\sigma)\frac{r}{t} = \frac{2}{|\nabla_y t|}$$

since $\nabla_y t = (2e_0 y', 1)$

$$\omega_n = -\tfrac{1}{2}(\nabla_y r)_n = -\tfrac{1}{2}(1+\sigma)\frac{r}{t} = -\frac{1}{|\nabla_y t|}$$

$$\Rightarrow \frac{x^n}{r} = \frac{y^n}{r} + \omega_n$$

$$< \frac{t}{r} - \frac{1}{|\nabla_y t|} \quad \text{b/c } t = M + y^n + e_0|y'|^2 > y^n$$

$$= \frac{1+\sigma}{2}|\nabla_y t| - \frac{1}{|\nabla_y t|}$$

$$< (1+\sigma)|\nabla_y t| - \frac{1}{|\nabla_y t|}$$

$$< c_0 \quad \text{by our previous choice of } \sigma, e_0 \text{ that makes } \omega + \nabla_y r \neq 0$$

$\Rightarrow \nabla_y r \neq -2\omega$ outside S

$\Rightarrow |\nabla_y r| \neq -2\omega$ outside S

In fact, $\nabla_y r + 2\omega$ bounded away from 0 outside S

Near $\nabla_y r = 0$: (aim for quadratic lower bound in nbd)

$$|\Pi_{\omega^\perp} \mathrm{Re}(U_1 - U_2)|^2 = \frac{|\Pi_{\omega^\perp} \nabla_y r|^2}{|\omega + \nabla_y r|^4}$$

$$= \frac{|\Pi_{\omega^\perp} \nabla_y r|^2}{(|1 + \nabla_y r \cdot \omega|^2 + |\Pi_{\omega^\perp} \nabla_y r|^2)^2}$$

$$\geq \frac{1}{2} |\Pi_{\omega^\perp} \nabla_y r|^2 \quad \text{near} \quad \nabla_y r = 0$$

$$|\Pi_{\omega^\perp} \mathrm{Im}(U_1 - U_2)|^2 = |\Pi_{\omega^\perp}(V_1 - V_2)|^2$$

$$\geq \big| |\Pi_{\omega^\perp} V_1| - |\Pi_{\omega^\perp} V_2| \big|^2$$

$$= \Big| 1 - \sqrt{\frac{1}{|\omega + \nabla_y r|^2} + |\omega \cdot V_2|^2} \Big|^2$$

$$= \Big| 1 - \sqrt{\frac{1}{1 + 2\omega \cdot \nabla_y r + |\nabla_y r|^2} + |\nabla_y r \cdot V_2|^2} \Big|^2$$

$$= |\omega \cdot \nabla_y r + O(|\nabla_y r|^2)|^2 \quad \text{b/c } |V_2| \approx 1 \text{ near } \nabla_y r = 0$$

$$= |\omega \cdot \nabla_y r|^2 + O(|\nabla_y r|^3)$$

$$\Rightarrow |\Pi_{\omega^\perp}(U_1 - U_2)| \geq \frac{1}{2} |\Pi_{\omega^\perp} \nabla_y r|^2 + |\omega \cdot \nabla_y r|^2 + O(|\nabla_y r|^3)$$

$$\geq \frac{1}{4} |\nabla_y r|^2 \quad \text{near} \quad \nabla_y r = 0$$

Thus, we may assume outside S, $|\Pi_{\omega^\perp}(U_1 - U_2)| \geq c_* |\nabla_y r|^2$ → Here c_* needed to be chosen INDEPENDENT of $\bar{\sigma}_0$, e

Next we show $D_y^2 r[U_1, U_2]$ bounded above

Indeed, $|D_y^2 r| = |(1+\sigma)\sigma \frac{r}{t^2} dt \otimes dt + (1+\sigma)\frac{r}{t} D_y^2 t|$

$$\leq (\sigma + 2e_3 \bar{\tau}_0) \frac{|\nabla_y r|^2}{r}$$

$$|U_2|^2 = \frac{2}{|\omega + \nabla_y r|^2} \leq \frac{2}{\delta^2} \quad \text{outside } S \text{ where } \delta \text{ chosen in terms of } c_0$$

$$\Rightarrow \left| r D_y^2 r [U_2, U_2] \right| \leq \frac{2}{\delta^2} (\sigma + 2\varepsilon_0 \bar{\tau}_0) |\nabla_y r|^2$$

Combine: $|\bar{\Pi}_{\omega^\perp}(U_1 - U_2)|^2 - r D_y^2 r [U_2, U_2] \geq \left[c_* - \frac{2}{\delta^2} (\sigma + 2\varepsilon_0 \bar{\tau}_0) \right] |\nabla_y r|^2$

> 0 b/c $\nabla_y r = (1+\sigma) a t^\circ \nabla t \neq 0$

by choosing σ, ε_0 suff small